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Background field boundary conditions for affine Toda field theories

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Abstract

Classical integrability is investigated for affine Toda field theories in the presence of a constant background tensor field. This leads to a further set of discrete possibilities for integrable boundary conditions depending on the time derivative of the fields at the boundary but containing no free parameters other than the bulk coupling constant.

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1. Introduction

There has been some interest in analysing the classical and quantum integrability of two-dimensional field theories with boundary, in which a theory is either restricted to a half-line, or to an interval. Some years ago, Cherednik and Sklyanin [1] developed a general mathematical machinery which generalised the standard tools applicable to integrable models to those cases in which a boundary condition needs to be taken into account. Principally, these tools are generalisations of the Yang-Baxter equations incorporating reflections from the boundary: the so-called reflection equations, and their classical counterparts.

Subsequently, Fring and Köberle, Ghoshal and Zamolodchikov, Sasaki, and Kim, [2-5], have developed a set of conjectures for the reflection factors of the sine-Gordon and affine Toda field theory models, based on a generalisation of the bootstrap idea. However, these conjectures are not easily related to specified boundary conditions[1]. If one merely asks for what boundary conditions is a field theory classically integrable, it might be expected there would be a class of boundary conditions introducing a set of boundary parameters in addition to the full-line parameters of the theory. Indeed, this is apparently the case for the sine-Gordon model where the most general integrable boundary condition contains two free parameters [3,6] [2] and is of the form:

$$\frac{\partial\phi}{\partial x^1} = \frac{a}{\beta}\sin\beta\left(\frac{\phi-\phi_0}{2}\right) \qquad \text{at } x^1=0, \tag{1.1}$$

where a and ϕ_0 are arbitrary constants, and β is the sine-Gordon coupling. However, and surprisingly, no other affine Toda field theory permits a full set of parameters (ie equal to the rank of the Lie algebra whose root data defines the model) in the boundary condition [9,7,10]. In fact, although the models based on non simply-laced algebras do allow some free parameters in the boundary condition, surprisingly few of the models permit boundary conditions continuously connected to the Neumann condition

$$\left.\frac{\partial\phi}{\partial x^1}\right|_{x^1=0} = 0. \tag{1.2}$$

Those that do are $c_n^{(1)}$, $a_{2n}^{(2)}$, $a_{2n-1}^{(2)}$, $d_4^{(3)}$ and $e_6^{(2)}$.

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[1] The exception to this is Kim's work in which conjectures are underpinned by perturbative arguments which can be carried out for the especially simple boundary condition $\partial_1\phi=0$.

[2] For reasons concerning stability these may not be chosen entirely arbitrarily [7,8].

It was supposed, in [9-10], that the boundary condition contained no time derivatives, but this could be too restrictive[3]: if we suppose there is no kinetic energy specifically associated with the boundary it is nevertheless possible, provided there is more than one scalar field, to envisage a boundary condition which is linear in time derivatives, in addition to being a function of the fields. In other words,

$$\frac{\partial \phi_a}{\partial x^1} = -b_{ab}\frac{\partial \phi_b}{\partial x^0} - \frac{\partial \mathcal{B}}{\partial \phi_a}, \qquad \text{at } x^1 = 0, \tag{1.3}$$

is a possible boundary condition, corresponding to an additional term in the Lagrangian of the form

$$-\delta(x^1)\left(\mathcal{B}(\phi) + \frac{1}{2}\phi_a b_{ab}\partial_0\phi_b\right), \tag{1.4}$$

where b_{ab} is an antisymmetric matrix. Such a boundary condition might be considered as the coupling of a (constant) background antisymmetric tensor field. Locally, this would be of the form

$$\partial_\mu\phi_a F^{\mu\nu}_{ab}\partial_\nu\phi_b,$$

a total derivative if each component of $F^{\mu\nu}$ satisfies free Maxwell equations. On integration, it would lead to the boundary term above with $b = F^{01}$. Boundary conditions of this general type have been considered recently for free fields by Yegulalp [11]. The two quantities b and $\mathcal{B}$ are to be determined by the requirements of integrability. It will be seen below that (1.3) with $b \neq 0$ is a rare possibility and is even more restrictive than the situation with $b = 0$. It will be seen there are no free parameters at all for these cases. Nevertheless, because of the lack of time-reversal invariance (1.3) can provide examples of classical reflection factors which differ between particle and anti-particle; such possibilities have been suggested previously by Sasaki on the basis of the reflection bootstrap equations [4].

In the case $b = 0$, it was found that the possible boundary conditions for the $a_n^{(1)}, d_n^{(1)}$ and $e_n^{(1)}$ affine Toda theories are highly constrained by the requirement that there should be conserved modifications of higher spin charges in the presence of the boundary. Effectively, in those cases, there is only a discrete ambiguity and the possible boundary conditions are summarised by adding a term to the action[4] of the form

$$\mathcal{L}_{\text{boundary}} = -\delta(x^1)\mathcal{B}(\phi), \tag{1.5}$$

[3] We are obliged to Nick Warner for reminding us of this.

[4] The notation and conventions for affine Toda field theory are those of [12]

where

$$\mathcal{B}=\frac{m}{\beta^2}\sum_0^r A_i e^{\frac{\beta}{2}\alpha_i\cdot\phi}, \tag{1.6}$$

and the coefficients $A_i,\ i=0,\dots,r$ are a set of real numbers with

$$\textbf{either } |A_i|=2\sqrt{n_i},\ \text{ for } i=0,\dots,r \textbf{ or } A_i=0 \text{ for } i=0,\dots,r\ . \tag{1.7}$$

This result is also obtained by generalising the Lax pair idea to include the boundary condition at $x^1=0$. Once the Lax pair is available, all the other cases can be investigated and are listed in [10].

To analyse the situation with $b\neq 0$, it is possible to proceed in two directions. Firstly, it is not difficult to repeat, case by case, the arguments of [9,7], construct conserved charges on the half-line using low-spin conserved charges defined for the whole line, and find constraints on the matrix b and the boundary potential $\mathcal{B}$. Alternatively, the Lax pair approach, mentioned earlier, can be adapted to the present situation and used as a tool to determine the possible choices of $b,\ \mathcal{B}$. In fact, in the case $b\neq 0$, the constraints are sufficiently severe that the latter approach turns out to provide, conveniently, a complete description of the classical problem.

2. Boundary Lax pair

The standard Lax pair for affine Toda theory [13] can be written in the form

$$\begin{aligned} a_0&=H\cdot\partial_1\phi/2+\sum_0^r\sqrt{m_i}(\lambda E_{\alpha_i}-1/\lambda\ E_{-\alpha_i})e^{\alpha_i\cdot\phi/2}\\ a_1&=H\cdot\partial_0\phi/2+\sum_0^r\sqrt{m_i}(\lambda E_{\alpha_i}+1/\lambda\ E_{-\alpha_i})e^{\alpha_i\cdot\phi/2}, \end{aligned} \tag{2.1}$$

where H_a, E_{α_i} and $E_{-\alpha_i}$ are the Cartan subalgebra and the generators corresponding to the simple roots, respectively, of a simple Lie algebra of rank r. Included in the set of 'simple' roots is the extra (affine) root, denoted α_0, which satisfies

$$\sum_0^r n_i\alpha_i=0 \qquad n_0=1.$$

The coefficients m_i are related to the n_i by $m_i=n_i\alpha_i^2/8$. The conjugation properties of the generators are chosen so that

$$a_1^\dagger(x,\lambda)=a_1(x,1/\lambda) \qquad a_0^\dagger(x,\lambda)=a_0(x,-1/\lambda). \tag{2.2}$$

Using the Lie algebra relations

$$[H, E_{\pm\alpha_i}] = \pm\,\alpha_i\, E_{\pm\alpha_i} \qquad [E_{\alpha_i}, E_{-\alpha_i}] = 2\alpha_i \cdot H/(\alpha_i^2),$$

the zero curvature condition for (2.1)

$$f_{01} = \partial_0 a_1 - \partial_1 a_0 + [a_0, a_1] = 0$$

leads to the affine Toda field equations:

$$\partial^2\phi = -\sum_0^r n_i \alpha_i e^{\alpha_i\cdot\phi}. \tag{2.3}$$

To construct a modified Lax pair including the boundary condition derived from (1.5), it was found in [10] to be convenient to consider an additional special point $x^1 = b\ (> a)$ and two overlapping regions $R_-:\ x^1 \le (a+b+\epsilon)/2;$ and $R_+:\ x^1 \ge (a+b-\epsilon)/2$. The second region will be regarded as a reflection of the first, in the sense that if $x^1 \in R_+$, then

$$\phi(x^1) \equiv \phi(a+b-x^1). \tag{2.4}$$

The regions overlap in a small interval surrounding the midpoint of $[a, b]$. Then, in the two regions define:

$$\begin{aligned} R_-: &\qquad \widehat{a}_0 = a_0 - \frac{1}{2}\theta(x^1-a)\left(\partial_1\phi + \frac{\partial\mathcal{B}}{\partial\phi}\right)\cdot H \qquad \widehat{a}_1 = \theta(a-x^1)a_1 \\ R_+: &\qquad \widehat{a}_0 = a_0 - \frac{1}{2}\theta(b-x^1)\left(\partial_1\phi - \frac{\partial\mathcal{B}}{\partial\phi}\right)\cdot H \qquad \widehat{a}_1 = \theta(x^1-b)a_1. \end{aligned} \tag{2.5}$$

Then, it is clear that in the region $x^1 < a$ the Lax pair (2.5) is the same as the old but, at $x^1 = a$ the derivative of the θ function in the zero curvature condition enforces the boundary condition

$$\frac{\partial\phi}{\partial x^1} = -\frac{\partial\mathcal{B}}{\partial\phi}, \qquad x^1 = a. \tag{2.6}$$

Similar statements hold for $x^1 \ge b$ except that the boundary condition at $x^1 = b$ is slightly different in order to accommodate the reflection condition (2.4).

On the other hand, for $x^1 \in R_-$ and $x^1 > a$, $\widehat{a}_1$ vanishes and therefore the zero curvature condition merely implies $\widehat{a}_0$ is independent of x^1. In turn, this fact implies ϕ is independent of x^1 in this region. Similar remarks apply to the region $x^1 \in R_+$ and $x^1 < b$. Hence, taking into account the reflection principle (2.4), ϕ is independent of x^1 throughout

the interval $[a, b]$, and equal to its value at a or b. For general boundary conditions, a glance at (2.5) reveals that the gauge potential $\widehat{a}_0$ is different in the two regions $R_\pm$. However, to maintain the zero curvature condition over the whole line the values of $\widehat{a}_0$ must be related by a gauge transformation on the overlap. Since $\widehat{a}_0$ is in fact independent of $x^1 \in [a, b]$ on both patches, albeit with a different value on each patch, the zero curvature condition effectively requires the existence of a gauge transformation $\mathcal{K}$ with the property:

$$\partial_0 \mathcal{K} = \mathcal{K}\, \widehat{a}_0(x^0, b) - \widehat{a}_0(x^0, a)\, \mathcal{K}. \tag{2.7}$$

The group element $\mathcal{K}$ lies in the group G with Lie algebra g, the Lie algebra whose roots define the affine Toda theory.

The conserved quantities on the half-line ($x \leq a$) are determined via a generating function $\widehat{Q}(\lambda)$ given by the expression

$$\widehat{Q}(\lambda) = \mathrm{tr}\left(U(-\infty, a; \lambda)\ \mathcal{K}\ U^\dagger(-\infty, a; 1/\lambda)\right), \tag{2.8}$$

where $U(x_1, x_2; \lambda)$ is defined by the path-ordered exponential:

$$U(x_1, x_2; \lambda) = \mathrm{P} \exp \int_{x_1}^{x_2} a_1 dx^1. \tag{2.9}$$

Assuming $\mathcal{K}$ is independent of both x^0 and the fields ϕ, or their derivatives, eq(2.7) reads,

$$\frac{1}{2}\left[\mathcal{K}(\lambda),\ \frac{\partial \mathcal{B}}{\partial \phi} \cdot H\right]_+ = -\left[\mathcal{K}(\lambda),\ \sum_0^r \sqrt{m_i}(\lambda E_{\alpha_i} - 1/\lambda\ E_{-\alpha_i}) e^{\alpha_i \cdot \phi/2}\right]_-, \tag{2.10}$$

where the field dependent quantities are evaluated at the boundary $x^1 = a$. Eq(2.10) is rather restrictive, since the boundary term $\mathcal{B}$ does not depend on the spectral parameter λ, and leads to the results concerning the boundary potential claimed in eqs(1.6) and (1.7), and given in detail elsewhere.

3. Modified boundary Lax pair

It is useful to note the Lax pair (2.1) is not unique. In particular, since the curvature f_{01} lies in the Cartan sub-algebra spanned by the generators H_a, the affine Toda equations

of motion will still be obtained if the 'gauge' fields a_0, a_1 are gauge transformed using any group element of the form

$$g = e^{i\theta\cdot H}.$$

However, this is not true of the boundary condition, coded via (2.5) in the modified fields $\widehat{a}_0, \widehat{a}_1$. Indeed, if the condition (2.6) is replaced by the condition (1.3), or its reflected version, in the definitions of $\widehat{a}_0$ in the two overlapping regions, then the additional terms proportional to $H_a b_{ab}\partial_0\phi_b$ can be removed by making a gauge transformation based on the group elements:

$$g_\pm = e^{\pm H\cdot b\phi} \tag{3.1}$$

in the regions $R_\pm$, respectively. After performing these gauge transformations, in the overlap region $a < x^1 < b$ one finds:

$$\begin{aligned}
\widehat{a}_0' &= -\frac{1}{2}\frac{\partial\mathcal{B}}{\partial\phi}\cdot H + \sum_0^r \sqrt{m_i}\left[\lambda e^{\alpha_i(1+b)\cdot\phi/2} - \frac{1}{\lambda}e^{\alpha_i(1-b)\cdot\phi/2}E_{-\alpha_i}\right] \quad x^1 \in R_- \\
\widehat{a}_0' &= \frac{1}{2}\frac{\partial\mathcal{B}}{\partial\phi}\cdot H + \sum_0^r \sqrt{m_i}\left[\lambda e^{\alpha_i(1-b)\cdot\phi/2} - \frac{1}{\lambda}e^{\alpha_i(1+b)\cdot\phi/2}E_{-\alpha_i}\right] \quad x^1 \in R_+.
\end{aligned} \tag{3.2}$$

These two, modified, gauge fields must then be related by $\mathcal{K}$ according to eq(2.7).

If it is further assumed $\mathcal{K}$ does not depend upon x^0 or ϕ, then (2.10) is modified to read:

$$\begin{aligned}
\frac{1}{2}\left[\mathcal{K}, \frac{\partial\mathcal{B}}{\partial\phi}\cdot H\right]_+ = -\sum_0^r \sqrt{m_i}\Bigg[&e^{\alpha_i(1-b)\cdot\phi/2}\left(\lambda\mathcal{K}E_{\alpha_i} + \frac{1}{\lambda}E_{-\alpha_i}\mathcal{K}\right) \\
&-e^{\alpha_i(1+b)\cdot\phi/2}\left(\frac{1}{\lambda}\mathcal{K}E_{-\alpha_i} + \lambda E_{\alpha_i}\mathcal{K}\right)\Bigg].
\end{aligned} \tag{3.3}$$

For the specific cases of interest, ϕ in (3.3) refers to the values of the field at $x^1 = 0$.

Next, suppose $\mathcal{K}(0)$ exists (at least after multiplying $\mathcal{K}$ by a suitable power of λ). Then, (3.3) reduces to

$$\sum_0^r \sqrt{m_i}\left[e^{\alpha_i(1+b)\cdot\phi/2}\mathcal{K}(0)E_{-\alpha_i} - e^{\alpha_i(1-b)\cdot\phi/2}E_{-\alpha_i}\mathcal{K}(0)\right] = 0, \tag{3.4}$$

and, since $\mathcal{K}$ is independent of the field value at the boundary, this in turn implies two conditions for each of $i = 0, \ldots, r$:

$$\alpha_i(1-b) = \alpha_{\pi(i)}(1+b) \tag{3.5}$$

where π is a permutation of $0, \ldots, r$ and,

$$\mathcal{K}(0)\sqrt{m_i}\, E_{-\alpha_i}\mathcal{K}^{-1}(0) = \sqrt{m_{\pi(i)}}\, E_{-\alpha_{\pi(i)}}. \tag{3.6}$$

The first of these conditions, (3.5), implies π is an automorphism of the extended Dynkin diagram whose root system defines the affine Toda theory under discussion (therefore $\sqrt{m_i} = \sqrt{m_{\pi(i)}}$); the second, (3.6), requires π to be an inner automorphism. In other words, π is a symmetry of the extended Dynkin diagram which is not also a symmetry of the Dynkin diagram itself—the group of such symmetries being isomorphic to the centre of the Lie group. From these observations, it is already very clear the conditions (1.3) are only rarely compatible with integrability. Indeed, the field theories which might allow integrable boundary conditions of this type may only be chosen from $a_r^{(1)}, d_r^{(1)}, e_6^{(1)}, e_7^{(1)}, b_r^{(1)}, c_r^{(1)}, a_{2r-1}^{(2)}, d_{r+1}^{(2)}$.

However, examining (3.5) carefully reveals that only odd order automorphisms are admissable. To see this, suppose π has order p and consider

$$\alpha_i(1-b) = \alpha_{\pi(i)}(1+b),\ \alpha_{\pi(i)}(1-b) = \alpha_{\pi^2(i)}(1+b),\ \ldots\ \alpha_{\pi^{p-1}(i)}(1-b) = \alpha_i(1+b), \tag{3.7}$$

and take the alternating sum to find

$$\alpha_i b = \alpha_{\pi(i)} - \alpha_{\pi^2(i)} +\ \ldots\ - \alpha_{\pi^{p-1}(i)}, \tag{3.8}$$

if p is odd and,

$$\alpha_{\pi(i)} - \alpha_{\pi^2(i)} +\ \ldots\ + \alpha_{\pi^{p-1}(i)} = 0, \tag{3.9}$$

when p is even. Adding (3.9) to a similar equation with i replaced by $\pi(i)$ immediately implies

$$\alpha_{\pi(i)} = -\alpha_i,$$

clearly impossible for a set of simple roots. Given the relevant automorphisms have odd order, the set of possible data is restricted to a choice from at most $a_r^{(1)}$ and $e_6^{(1)}$.

Since $\mathcal{K}(0)$ represents an inner automorphism of the Lie algebra of a compact Lie group, choose it to be unitary. Then, using (3.6) it follows that

$$\mathcal{K}(0)\alpha_i \cdot \mathcal{K}^{-1}(0) = \alpha_{\pi(i)} \cdot H. \tag{3.10}$$

Setting

$$\mathcal{K}(\lambda) = (1 + k_1\lambda + k_2\lambda^2 + O(\lambda^3))k_0, \tag{3.11}$$

and examining the order λ terms in (3.3), leads to an equation determining both k_1 and the boundary potential $\mathcal{B}$:

$$\begin{aligned}
&\frac{1}{2}\frac{\partial \mathcal{B}}{\partial \phi}\cdot\left(H+\ k_0 H k_0^{-1}\right)\\
&\qquad\qquad = \sum_0^r \sqrt{m_i}\left[e^{\alpha_i(1+b)\cdot\phi/2}\,k_1\,k_0 E_{-\alpha_i}k_0^{-1} - e^{\alpha_i(1-b)\cdot\phi/2}E_{-\alpha_i}\,k_1\right]\\
&\qquad\qquad = \sum_0^r \sqrt{m_i}e^{\alpha_i(1-b)\cdot\phi/2}[k_1,\,E_{-\alpha_i}],
\end{aligned}\tag{3.12}$$

the second step following immediately from (3.5) and (3.6). Clearly, considering the grading of the Lie algebra generators on the two sides of (3.12), bearing in mind (3.10), k_1 must be a linear combination of the positive simple root step operators and the step operator corresponding to the lowest root α_0:

$$k_1 = \sum_0^r (A_i/\sqrt{m_i})E_{\alpha_i},\tag{3.13}$$

where the A_i are a set of constants. Therefore, using (3.10) and (3.5), (3.12) reduces to an equation constraining $\mathcal{B}$

$$\frac{1}{2}\frac{\partial \mathcal{B}}{\partial \phi}\cdot\big(H+(1+b)(1-b)^{-1}H\big)=\sum_0^r\,\alpha_i\cdot He^{\alpha_i(1-b)\cdot\phi/2}.\tag{3.14}$$

Matching the coefficients of the independent elements of the Cartan subalgebra, and multiplying through by $1-b$ yields

$$\frac{\partial \mathcal{B}}{\partial \phi}=\sum_0^r A_i\alpha_i(1-b)e^{\alpha_i(1-b)\cdot\phi/2},$$

which implies

$$\mathcal{B}=2\sum_0^r A_i e^{\alpha_i(1-b)\cdot\phi/2}.\tag{3.15}$$

Thus, provided the coefficients A_i are not subsequently forced to vanish, the boundary potential again has the characteristic exponential form although the exponents may contain vectors other than simple roots. If $b=0$, (3.15) reduces to the results obtained before [7,9,10].

Using the expression for $\mathcal{B}$, and assuming $\mathcal{K}$ is independent of ϕ, leads to a set of equations, one for each $i = 0, 1, \ldots, r$, from which any further constraints on $\mathcal{K}(\lambda)$ will be derived:

$$\begin{aligned}\frac{A_i}{2}\,[\mathcal{K}(\lambda),\,\alpha_i(1-b)\cdot H]_+ = &-\lambda\,\mathcal{K}(\lambda)\,E_{\alpha_i} + \frac{1}{\lambda}\,\mathcal{K}(\lambda)\,E_{-\alpha_{\pi^{-1}(i)}} \\ &+\lambda\,E_{\alpha_{\pi^{-1}(i)}}\,\mathcal{K}(\lambda) - \frac{1}{\lambda}\,E_{-\alpha_i}\,\mathcal{K}(\lambda).\end{aligned} \tag{3.16}$$

To analyse further the generic set of cases, $a_r^{(1)}$, it is convenient to work in the fundamental representation and to introduce the following pair of matrices P and Q,

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 \\ 0 & 0 & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix} \qquad Q = \mathrm{Diag}(1, \omega, \omega^2, \omega^3, \ldots, \omega^r), \quad \omega^{r+1} = 1, \tag{3.17}$$

which satisfy

$$P^{r+1} = Q^{r+1} = 1, \qquad PQ = \omega QP. \tag{3.18}$$

In terms of these, the generators corresponding to the simple roots are given by

$$E_{\alpha_k} = \frac{1}{r+1}\sum_{s=0}^{r} \omega^{-ks} PQ^s, \qquad k = 0, 1, \ldots, r, \tag{3.19}$$

and the elements of the Cartan subalgebra are

$$\alpha_k \cdot H = \frac{1}{r+1}\sum_{s=0}^{r} (\omega^s - 1)\omega^{-ks} Q^s, \qquad k = 0, 1, \ldots, r. \tag{3.20}$$

Using (3.18) and (3.19), it is easy to check P implements an elementary cyclic permutation of the generators corresponding to the simple roots:

$$\begin{aligned} P^{-1} E_{\alpha_k} P &= \frac{1}{r+1}\sum_{s=0}^{r} \omega^{-ks} P \omega^{-s} Q^s = E_{\alpha_{k+1}} \\ P^{-1}\alpha_k \cdot HP &= \alpha_{k+1}\cdot H, \end{aligned} \tag{3.21}$$

and, therefore, the set $P^s,\ s = 0, 1, 2, \ldots, r$ are the elements of the Z_{r+1} group of symmetries of the $a_r^{(1)}$ extended Dynkin-Kač diagram.

Suppose the permutation π in eq(3.16) is represented by P^L then, using (3.19) and (3.20), eq(3.16) may be rewritten usefully as follows:

$$\begin{aligned}\sum_s \omega^{-ks}\Bigg(&\frac{A_k}{2}(\omega^s - 1)(1 - \omega^{Ls} + \omega^{2Ls} - \ldots + \omega^{(p-1)Ls})\,[\mathcal{K}, Q^s]_+ \\ &+\lambda\mathcal{K}PQ^s - \frac{1}{\lambda}\omega^{-Ls}\mathcal{K}Q^sP^{-1} - \lambda\omega^{-Ls}PQ^s\mathcal{K} + \frac{1}{\lambda}Q^sP^{-1}\mathcal{K}\Bigg) = 0.\end{aligned} \tag{3.22}$$

Given the form of eq(3.22), it seems natural to take, as an ansatz for $\mathcal{K}$,

$$\mathcal{K}(\lambda) = \sum_t k_t(\lambda) P^t, \tag{3.23}$$

and to suppose the coefficients of $\omega^{-ks} Q^s P^t$ vanish in eq(3.22) for each choice of k, s and t. In other words,

$$\begin{aligned} &\frac{A_k}{2}(\omega^s - 1)(1+\omega^{ts})(1-\omega^{Ls}+\omega^{2Ls}-\ldots+\omega^{(p-1)Ls})k_t \\ &\qquad + \lambda k_{t-1}\omega^{ts} - \lambda k_{t-1}\omega^{(1-L)s} - \frac{1}{\lambda}k_{t+1}\omega^{(t+1-L)s} + \frac{1}{\lambda}k_{t+1} = 0, \end{aligned} \tag{3.24}$$

where the coefficients k_t depend upon t and λ but do not depend upon s or k. Indeed, the only dependence on the label k occurs in the coefficients A_k which must therefore be equal to each other (they may all be zero). Apart from the latter remark, there are several cases.

I: $A_k = 0 \;\; k = 0, 1, \ldots, r$

In this case, (3.24) reduces to

$$k_{t+1}(1-\omega^{(t-L+1)s}) = \lambda^2 k_{t-1}(\omega^{(1-L)s} - \omega^{ts}),$$

which has a solution provided $L = -1$ (and, therefore, r must be even), and all but two of the coefficients (k_1 and k_r) are zero. Ie $\mathcal{K}$ is given by

$$\mathcal{K}(\lambda) = \lambda P - \frac{1}{\lambda}P^{-1}, \tag{3.25}$$

and b is given by (3.8), with $\pi = P^{-1}$.

II: $A_0 = A_1 = A_2 = \ldots = A_r = A$

In this case, (3.24) may be reorganised by multiplying through by $1+\omega^{Ls}$, to obtain:

$$\begin{aligned} &A(\omega^s - 1)(1+\omega^{ts})k_t \\ &+ (1+\omega^{Ls})\left(\lambda k_{t-1}\omega^{ts} - \lambda k_{t-1}\omega^{(1-L)s} - \frac{1}{\lambda}k_{t+1}\omega^{(t+1-L)s} + \frac{1}{\lambda}k_{t+1}\right) = 0, \end{aligned}$$

which is solved (assuming none of the coefficients vanish) provided

$$Ak_t = \lambda k_{t-1} = \frac{1}{\lambda}k_{t+1}, \qquad \omega^{2Ls} = \omega^s,$$

in turn implying

$$A^2 = 1, \qquad L = \frac{r+2}{2}. \tag{3.26}$$

Again, r must be even, and $\mathcal{K}$ is given by

$$\mathcal{K}(\lambda) = \sum_{-r/2}^{r/2} (A\lambda P)^t. \tag{3.27}$$

III: $r = 5$

Special cases may occur in II if some of the coefficients k_t in fact vanish. However, direct inspection reveals there is precisely one such case for which

$$r = 5, \qquad L = -2, \qquad A^2 = 1, \tag{3.28}$$

and

$$\mathcal{K}(\lambda) = \frac{1}{\lambda^2} P^{-2} - \frac{A}{\lambda} P^{-1} + A\lambda P - \lambda^2 P^2. \tag{3.29}$$

For this, the permutation π is of order three.

To analyse the remaining case, $e_6^{(1)}$, return to the perturbative expansion of $\mathcal{K}$, eq(3.11), and attempt to determine the term at order λ^2, given (3.25) and (3.15). After some manipulation, one obtains the set of equations:

$$\begin{aligned}[k_2, E_{-\alpha_i}] = E_{\alpha_{\pi(i)}} &- E_{\alpha_{\pi^{p-1}(i)}} \\ &- \frac{A_i}{2} \left(\alpha_{\pi(i)} - \alpha_{\pi^2(i)} + \ldots - \alpha_{\pi^{p-1}(i)}\right) \cdot \sum_j A_j \alpha_j E_{\alpha_j},\end{aligned} \tag{3.30}$$

for each of $i = 0, 1, \ldots, r$. Clearly, k_2 must be a linear combination of generators corresponding to level two roots. The permutation π is the threefold symmetry of the extended $e_6^{(1)}$ diagram whose orbits consist of the three outer roots (labelled 0,1,2) the three inner roots (labelled 3,4,5) and the centre root (labelled 6), taken clockwise with the pairs (0,3), (1,4) and (2,5) lying on the three legs, respectively. Examining, eqs(3.30) for $i = 0, 1, 2$ leads immediately to the conclusion $A_3 = A_4 = A_5 = 0$. However, the equations corresponding to i=3,4,5 are then inconsistent; for example, when $i = 3$

$$[k_2, E_{-\alpha_3}] = E_{\alpha_4} - E_{\alpha_5}$$

which may never be satisfied since $\alpha_4 + \alpha_3$ and $\alpha_5 + \alpha_3$ are not roots. Hence, for $e_6^{(1)}$ the hypothesis concerning the existence of $\mathcal{K}(0)$ is false, and $\mathcal{K}$ cannot exist.

4. Discussion

The linearised version of the field equations and the boundary conditions may be examined, noting that in all the allowable cases I, II and III, $\phi^{(0)} = 0$ is a possible classical solution.

First, note that the matrix b and the mass matrix of the affine Toda field theory commute. The mass matrix M is defined by

$$M^2 = \sum_0^r n_i \alpha_i \otimes \alpha_i, \tag{4.1}$$

and therefore, using the antisymmetry of b, the commutator

$$\left[M^2, b\right] = \sum_0^r n_i \left(\alpha_i \otimes \alpha_i b + \alpha_i b \otimes \alpha_i\right), \tag{4.2}$$

can be evaluated using (3.8). Rearranging the sums using the permutation π, and recalling π has odd order, leads to the terms on the right hand side of (4.2) cancelling pairwise to zero. Actually, this fact was to be expected since the mass-matrix is invariant under permutations of the roots corresponding to symmetries of the Dynkin, or extended Dynkin-Kač, diagram and b is directly related to such a symmetry.

Second, to express b in terms of π, consider the latter as a linear mapping of the roots:

$$\alpha_{\pi(i)} = \widehat{\pi}\alpha_i, \tag{4.3}$$

and use eq(3.7), to deduce:

$$b = \frac{\widehat{\pi} - 1}{\widehat{\pi} + 1}, \tag{4.4}$$

from which, knowing the eigenvalues of $\widehat{\pi}$, the eigenvalues of b may be read off. In every case, $\widehat{\pi}$ is a power of the permutation matrix which, acting on the roots, has eigenvalues which are the $(r+1)$st roots of unity, except 1 itself. Hence the eigenvalues of b, $b_s,\ s = 1, 2, \ldots, r$, are

$$b_s = i \tan\left(\frac{\pi L s}{r+1}\right) = -b_{r+1-s}. \tag{4.5}$$

Case I is the simplest to treat because there is no boundary potential. There is a scattering solution to the linearised problem, of the form:

$$\phi^{(1)} = \epsilon \sum_s \rho_s e^{-i\omega_s x^0} \left(R_s e^{-ik_s x^1} + e^{ik_s x^1}\right) \qquad x^1 < 0, \tag{4.6}$$

where ρ_s are the common eigenvectors of the mass^2 matrix and the boundary matrix b:

$$M^2\rho_s = m_s^2\rho_s \qquad b\rho_s = b_s\rho_s,$$

and

$$\omega_s = m_s\cosh\theta \qquad k_s = m_s\sinh\theta.$$

Using (4.5), with $L=-1$, and the boundary condition (1.3), yields an expression for the reflection factors R_s:

$$R_s = \frac{k_s - b_s\omega_s}{k_s + b_s\omega_s} = -\frac{(s)}{(r+1-s)}, \tag{4.7}$$

where the final step uses the notation

$$(x) = \frac{\sinh\left(\frac{\theta}{2} + \frac{i\pi x}{2(r+1)}\right)}{\sinh\left(\frac{\theta}{2} - \frac{i\pi x}{2(r+1)}\right)}.$$

Notice, as a consequence of (4.5), the reflection factors for a particle and its conjugate are not equal; rather $R_{\bar{s}} = R_s^{-1} = R_s(i\pi - \theta)$. Clearly, a distinction between the two classical reflection factors was to be expected since the boundary condition is not time-reversal invariant. Some time ago, Sasaki [4] discovered asymmetric solutions to the reflection bootstrap equations but without noting examples of boundary conditions which might be responsible for them. Notice also, these reflection factors satisfy the classical limit of the reflection bootstrap equations for the $a_r^{(1)}$ theories. In other words, if the three particles r, s, t couple in the quantum field theory (in the sense that any of them, say r, may be a bound state of the other two, s and t) then, assuming factorisability, the reflection bootstrap equations provide an expression for K_r in terms of K_s and K_t:

$$K_r(\theta) = K_s(\theta - i\bar{\theta}_{sr}^t)K_t(\theta + i\bar{\theta}_{tr}^s)S_{st}(2\theta). \tag{4.8}$$

The classical limit of (4.8) replaces K by R and S by unity. The notation, and the data concerning coupling angles necessary to verify the assertion can be found in [12].

For cases II and III, the linearised boundary condition is more complicated and for each mass eigenvalue takes the form

$$\partial_1\phi_s = -b_s\partial_0\phi_s - \frac{A}{2}(1-b_s^2)m_s^2\phi_s \qquad x^1 = 0. \tag{4.9}$$

For example, in case III, $b_s = -i\tan(\pi s/3)$ and the linearised reflection factors are given by

$$R_s = \frac{\sinh\theta\cos^2(\pi s/3) + i\cosh\theta\cos(\pi s/3)\sin(\pi s/3) + iA\sin(\pi s/6)}{\sinh\theta\cos^2(\pi s/3) - i\cosh\theta\cos(\pi s/3)\sin(\pi s/3) - iA\sin(\pi s/6)}. \tag{4.10}$$

Curiously, taking $A = 1$ for instance, one finds $R_1 = (5)^2$ in the standard notation but, the classical version of the reflection bootstrap equation is not actually satisfied since $R_2 \neq (4)^2$. In this instance, it appears the solution to the linear problem is not the classical limit of a quantum theory with a factorisable S-matrix and reflection factors; perhaps the quantum field theory is not integrable in these cases. Similar remarks apply to reflection factors derived in case II.

It appears classical integrability in the presence of boundary conditions is a rare phenomenon and it is curious that very few of the known examples (apart from sinh/sine-Gordon) permit either a continuous deformation away from the Neumann condition, $\partial_1\phi\big|_{x^1=0} = 0$, or a continuous deformation away from the free or from the conformal Toda situation. It seems a 'quantisation' of the boundary parameters is largely inescapable.

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Appendix

There is a direct argument implying that the classical reflection factors should satisfy a bootstrap condition. Since the argument does not seem to appear elsewhere in the literature, it will be included in outline here for completeness.

Suppose $\phi^{(0)} = 0$ is a solution to the full field equations, plus a boundary condition at $x^1 = 0$. Then, assuming there is another solution which may be considered to be small, it has an expansion (in terms of the coupling constant if one prefers), of the type:

$$\phi = \phi^{(1)} + \phi^{(2)} + \dots\ .$$

The first two terms satisfy the equations:

$$\begin{aligned}(\partial^2 + M^2)_{ab}\phi_b^{(1)} &= 0\\ (\partial^2 + M^2)_{ab}\phi_b^{(2)} &= -c^{abc}\phi_b^{(1)}\phi_c^{(2)},\end{aligned} \tag{4.11}$$

where c^{abc} are the classical couplings to be found in [12]. With a boundary, the solutions sought are perturbations of the solutions to the linear equation given in (4.6). Thus,

$$\begin{aligned}\phi^{(2)} = \epsilon^2 \sum_{r,s,t} \rho_r\, \widehat{c}_{rst}\, e^{-i(\omega_s+\omega_t)x^0} \times \\ \Bigg(\frac{1}{(\omega_s+\omega_t)^2-(k_s+k_t)^2-m_r^2}\left[e^{i(k_s+k_t)x^1}+R_sR_te^{-i(k_s+k_t)x^1}\right] \\ + \frac{1}{(\omega_s+\omega_t)^2-(k_s-k_t)^2-m_r^2}\left[R_te^{i(k_s-k_t)x^1}+R_se^{-i(k_s-k_t)x^1}\right]\Bigg),\end{aligned} \tag{4.12}$$

where

$$\widehat{c}_{rst} = c^{abc}\rho_r^a\rho_s^b\rho_t^c,$$

and it has been assumed the eigenvectors ρ_s^a are normalised to unity.

Clearly, the first term on the right hand side of (4.12) has a pole when the momenta and energy of particles s and t happen to lie on the mass-shell of particle r, and the term exists in the sum for a particular s and t with a classical coupling to r. If this can happen, the term $\phi^{(2)}$ dominates and consistency with the boundary conditions would require the coefficient of the pole to agree with the leading order reflection coefficient of particle r. In other words, one is led to deduce a classical bootstrap condition

$$K_r(\theta) = K_s(\theta - i\bar{\theta}^t_{sr})K_t(\theta + i\bar{\theta}^s_{tr}),$$

reminiscent of the bootstrap property of the soliton solutions in the complex affine Toda field theory [14]. Apparently, the classical bootstrap property depends only on the field equations in the region $x < 0$. The difficulty with this argument rests with the boundary condition. The first order approximation has been designed to satisfy the boundary condition at $x = 0$ but there is no guarantee that the next order term, determined by eq(4.12), will do so. In general, it may not and an extra term of order ϵ^2, satisfying the homogeneous equation, must be added to (4.12) to maintain the boundary condition.

Returning to the two examples given in section 4, the first, for which the reflection coefficient is given by (4.7), leads to a second order term (4.12) which does in fact satisfy the boundary condition automatically. Although it has not been checked beyond the second order, one suspects the (minimal) perturbative solution satisfies the boundary condition order-by-order in this case. On the other hand, the second example, for which the reflection coefficient is given by (4.10), does not lead to a perturbative solution satisfying the boundary condition without the explicit addition of extra pieces at each order.

References

[1] I. V. Cherednik, 'Factorizing particles on a half line and root systems', *Theor. Math. Phys.* **61** (1984) 977;
E. K. Sklyanin, 'Boundary conditions for integrable equations', *Funct. Analy. Appl.* **21** (1987) 164;
E. K. Sklyanin, 'Boundary conditions for integrable quantum systems', *J. Phys.* **A21** (1988) 2375.

[2] A. Fring and R. Köberle, 'Factorized scattering in the presence of reflecting boundaries', *Nucl. Phys.* **B421** (1994) 159;
A. Fring and R. Köberle, 'Affine Toda field theory in the presence of reflecting boundaries', *Nucl. Phys.* **B419** (1994) 647.

[3] S. Ghoshal and A.B. Zamolodchikov, 'Boundary S matrix and boundary state in two-dimensional integrable quantum field theory', *Int. J. Mod. Phys.* **A9** (1994) 3841.

[4] R. Sasaki, 'Reflection bootstrap equations for Toda field theory', in *Interface between Physics and Mathematics*, eds W. Nahm and J-M Shen, (World Scientific 1994) 201.

[5] J.D. Kim, 'Boundary reflection matrix in perturbative quantum field theory', *Phys. Lett.* **B353** (1995) 213;
J.D. Kim, 'Boundary reflection matrix for A-D-E affine Toda field theory', DTP-95-31; hep-th/9506031.

[6] A. MacIntyre, 'Integrable boundary conditions for classical sine-Gordon theory', *J. Phys.* **A28** (1995) 1089.

[7] E. Corrigan, P.E. Dorey, R.H. Rietdijk, 'Aspects of affine Toda field theory on a half line', to appear in Proceedings of 'Quantum field theory, integrable models and beyond', Yukawa Institute for Theoretical Physics, Kyoto University, February 1994; hep-th/9407148.

[8] A. Fujii and R. Sasaki, 'Boundary effects in integrable field theory on a half line', YITP-K-1091; hep-th/9503083.

[9] E. Corrigan, P.E. Dorey, R.H. Rietdijk and R. Sasaki, 'Affine Toda field theory on a half line', *Phys. Lett.* **B333** (1994) 83.

[10] P. Bowcock, E. Corrigan, P.E. Dorey and R. H. Rietdijk, 'Classically integrable boundary conditions for affine Toda field theories', *Nucl. Phys.* **B445** (1995) 469.

[11] A. Yegulalp, 'New boundary conformal field theories indexed by the simply-laced Lie algebras', *Nucl. Phys.* **B450** (1995) 641.

[12] H. W. Braden, E. Corrigan, P.E. Dorey and R. Sasaki, 'Affine Toda field theory and exact S-matrices', *Nucl. Phys.* **B338** (1990) 689.

[13] A. V. Mikhailov, M. A. Olshanetsky and A. M. Perelomov, 'Two-dimensional generalised Toda lattice', *Commun. Math. Phys.* **79** (1981) 473;

G. Wilson, 'The modified Lax and and two-dimensional Toda lattice equations associated with simple Lie algebras', *Ergod. Th. and Dynam. Sys.* **1** (1981) 361;
D. I. Olive and N. Turok, 'The symmetries of Dynkin diagrams and the reduction of Toda field equations', *Nucl. Phys.* **B215** (1983) 470.

[14] T.J. Hollowood, 'Solitons in affine Toda field theories', *Nucl. Phys.* **384** (1992) 523.